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The approximation of a Crank–Nicolson scheme for the stochastic Navier–Stokes equations☆

Xiaoyuan Yang*, Wei Wang, Yuanyuan Duan

Department of Mathematics, Beihang University, LMIB of the Ministry of Education, Beijing 100083, China

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ABSTRACT

In this paper we prove the convergence of stochastic Navier–Stokes equations driven by white noise. A linearized version of the implicit Crank–Nicolson scheme is considered for the approximation of the solutions to the N–S equations. The noise is defined as the distributional derivative of a Wiener process and approximated by using the generalized L_2-projection operator. Optimal strong convergence error estimates in the L_2 norm are obtained.

1. Introduction

The Navier–Stokes equation has important physical and technical applications. It describes the behavior of a viscous velocity field of an incompressible liquid. The Galerkin method is one of the well-known methods in the theory of partial differential equations that is used to prove existence properties and to obtain finite dimensional approximations for the solutions of the equations. For the stochastic Navier–Stokes equation, the existence, the uniqueness, and the properties of the solutions of such equations have been well studied. However, Galerkin approximation for stochastic Navier–Stokes equation has not been fully researched.

The Navier–Stokes equations were first formulated by the French physicist Navier in 1822 and the British mathematician and physicist G.G. Stokes in 1845. Existence and uniqueness theorems for the stationary Navier–Stokes equation were first proved by F. Odquist in 1930 and J. Leray. Hopf [1] was the first who obtained the equation for the characteristic functional of the statistical solution giving a probability description of fluid flows. C. Foias investigated in [2] the questions of existence and uniqueness of spatial statistical solutions. Bensoussan and Temam [3] gave for the first time a functional analytical approach for the stochastic Navier–Stokes equations. Important results concerning the theory and numerical analysis of the deterministic Navier-stokes equation can be found in the book of Temam [4]. Breckner [5] proved that the error of numerical approximation of a stochastic Navier–Stokes equation converges to zero when the time step of the scheme is sufficiently small.

The Galerkin method can be used for the stochastic Navier–Stokes equation. Bensoussan [6], Capinski, Cutland [7], Gatarek [8], Komech, Vishik [9], and M. Viot gave us the examples about how to use the Galerkin method to investigate the stochastic Navier–Stokes equation. They have considered the weak solutions. The techniques used in the proofs were the construction of the Galerkin-type approximations of the solutions and some a priori estimates, that allowed one to prove compactness properties of the corresponding probability measures and finally to obtain a solution of the equation. In the papers of Grecksch, Kloeden [11], Gyöngy [12], and E. Pardoux, the authors investigated evolution equations with Lipschitz

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* Corresponding author.
E-mail address: xiaoyuanyang@vip.163.com (X. Yang).

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continuous nonlinearities. Yubin Yan [10] gave the error estimates for the stochastic parabolic partial differential equations based on appropriate nonsmooth data error estimates for the corresponding deterministic parabolic problem.

In this paper we consider the approximation of strong solution of the stochastic Navier–Stokes equation which needs different methods to prove, but we also use the idea of adapting the deterministic Galerkin method to the stochastic case. First, we take the stochastic Navier–Stokes equation in the generalized sense as an evolution equation, assuming that the stochastic processes are defined on a given complete probability space. The noise is defined as the distributional derivative of a Wiener process and approximated by using the generalized L_2-projection operator. The aim of this paper is to prove the error estimates of the solution of the Navier–Stokes equation by approximating it in the L_2 norm. Therefore, in order to obtain the main estimation we firstly device the filtered probability space and define the L_2 space and its norm. Under the L_2 norm, we derive the isometry property about the expectation and some inequalities about the operator. Some estimates for the corresponding deterministic problem that are needed for the main result are proved in the Lemmas. The error between the exact solution and the numerical solution of the stochastic equation is split into different parts, and we deal with these errors in different ways. Finally, we use the properties of stopping times and some basic convergence principles to complete the proof of the error estimates of the stochastic Navier–Stokes equation.

The structure of the paper is as follows: in Section 2 we introduce the notations of the probability spaces and some useful operators including their property. We also give some preliminary results in order to prove the main theorem. In Section 3 we complete the proof for the main results and some relative lemmas. Finally Section 4 is our conclusion of this paper.

2. Notations and preliminaries

We consider the Navier–Stokes equations on a finite time interval $[0, T]$. Let D be a connected and bounded subset of R^2, with a regular enough boundary ∂D. Consider the following stochastic Navier–Stokes equations:

$$\frac{\partial u}{\partial t}(x,t) - \nu\Delta u(x,t) + (u\cdot\nabla)u(x,t) + \nabla p(x,t) = \sigma(u(x,t))\frac{\partial W}{\partial t}, \quad x\in D, t>0, \tag{2.1}$$

and the continuity equations:

$$\operatorname{div} u(x,t) = 0, \quad x\in D, t>0, \tag{2.2}$$

where u is the velocity field, ν is the viscosity, Δ is the Laplacian, ∇ is the gradient, p is the pressure. The right hand term $\sigma(u)\frac{\partial W}{\partial t}$ describes a state dependent random noise.

The initial boundary conditions are:

$$u(x,t) = 0 \quad x\in\partial D,\ t>0 \tag{2.3}$$

and

$$u(x,0) = u_0(x) \quad x\in D,\ t=0 \tag{2.4}$$

Now we introduce some notations and assumptions about the stochastic N–S equation to be considered.

Let (V, H, V^*) be an evolution triple, where $(V, \|\cdot\|_V)$ and $(H, \|\cdot\|)$ are separable Hilbert spaces, and the embedding operator $V \hookrightarrow H$ is assumed to be compact. V^* is the dual space of V. $(\cdot,\cdot)$ denotes the inner product in H.

We define $A : V \to V^*$ as a linear, bounded, self-adjoint, positive definite operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, such that $\langle Au, u\rangle \ge C\|u\|_V^2$ for all $u \in V$ and $\langle Au, v\rangle = \langle Av, u\rangle$ for all $u, v \in V$, where $C > 0$ is a constant and $\langle\cdot,\cdot\rangle$ denotes the dual pairing.

$B : V\times V \to V^*$ is bilinear operator such that

$$(B(u,v), v) = 0, \quad \forall u, v\in V, \tag{2.5}$$

and for which there exists a constant $C > 0$ such that

$$|(B(u,v), w)|^2 \le C\|u\|\|u\|_V\|v\|\|v\|_V\|w\|_V^2, \quad \forall u, v, w\in V. \tag{2.6}$$

$\sigma : V \to H$ is operator-valued function defined on H such that

$$\|\sigma(u)-\sigma(v)\|^2 \le C\|u-v\|^2. \tag{2.7}$$

$(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ is a right-continuous filtration such that $\mathcal{F}_0$ contains all $\mathcal{F}$-null sets. $(W(t))_{t\in[0,T]}$ denotes a real valued standard $\mathcal{F}_t$-Wiener process.

Now we can let P denote the orthogonal projection, $Au = -P\Delta u$ and $B(u,v) = P(u\cdot\nabla)v$. Within this framework, (2.1) and (2.2) are expressed as the evolution equation:

$$\frac{\partial u}{\partial t} + Au + B(u,u) = \sigma(u)\frac{\partial W}{\partial t}, \quad u\in H, t>0, \tag{2.8}$$

Let $\mathbf{E}$ denote the expectation. An adapted V-valued process $(u(t))_{t\in[0,T]}$ with $\mathbf{E}\|u(t)\|^2 < \infty$ for all $t \in [0, T]$ and $\mathbf{E}\int_0^T \|u(t)\|_V^2 \mathrm{d}t < \infty$ is called a strong solution of the stochastic Navier–Stokes equation if it satisfies the equation:

$$(u(t), v) + \int_0^t (Au(s), v)\, \mathrm{d}s + \int_0^t (B(u(s), u(s)), v)\mathrm{d}s = (u_0, v) + \int_0^t (\sigma(u(s)), v)\mathrm{d}W(s) \tag{2.9}$$

for all $v \in V$, $t \in [0, T]$ and for $a.e.\ \omega \in \Omega$, where the stochastic integral is understood in the Itô sense. As usual, in the notation of random variables or stochastic processes we generally omit the dependence on ω.

For any Hilbert space H, we define

$$L_2(\Omega; H) = \left\{ v : \mathbf{E}\|v\|_H^2 = \int_\Omega \|v(\omega)\|_H^2\, \mathrm{d}\mathbf{P}(\omega) < \infty \right\}, \tag{2.10}$$

with norm $\|v\|_{L_2(\Omega;H)} = (\mathbf{E}\|v\|_H^2)^{1/2}$.

Let $v \in L_2(\Omega; H)$. Then $\int_0^t v(s)\,\mathrm{d}s$ can be defined and the following isometry property holds:

$$\mathbf{E}\left\| \int_0^t v(s)\mathrm{d}W(s) \right\|^2 = \int_0^t \|\mathbf{E}v(s)\|^2 \mathrm{d}s. \tag{2.11}$$

Let $E(t) = \mathrm{e}^{-tA}$, $t > 0$. Then (2.8) admits a unique mild solution (see [10]):

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\sigma(u)\mathrm{d}W(s) - \int_0^t E(t-s)B(u, u)\, \mathrm{d}s \tag{2.12}$$

where $E(t)$ is the analytic semigroup generated by $-A$.

Let τ be a time step and $t_n = n\tau$ with $n > 1$. We apply a linearized version of Crank–Nicolson scheme to (2.8):

$$\frac{u^n - u^{n-1}}{\tau} + A\frac{u^n + u^{n-1}}{2} + B\left(u^{n-1}, \frac{u^{n-1} + u^n}{2}\right) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \frac{\sigma(u^n) + \sigma(u^{n-1})}{2}\mathrm{d}W(s), \quad \text{for } n > 1,\ u^0 = u_0 \tag{2.13}$$

which determines a sequence of $u^n \in D(A),\ n = 1, 2, \ldots$.

First we recall a discrete version of the Uniform Gronwall lemma [13] which will be useful in our discussion.

Lemma 2.1. *Let τ, B, and a_j, b_j, c_j, γ_j, for integers $j \ge 0$, be nonnegative numbers such that*

$$a_n + \tau\sum_{j=0}^n b_j \le \tau\sum_{j=0}^n \gamma_j a_j + \tau\sum_{j=0}^n c_j + B \quad \text{for } n \ge 0.$$

Suppose that $\tau\gamma_j < 1$ for all j, then

$$a_n + \tau\sum_{j=0}^n b_j \le \mathrm{e}^{\tau\sum_{j=0}^n \frac{\gamma_j}{1-k\gamma_j}} \left(\tau\sum_{j=0}^n c_j + B\right) \quad \text{for } n \ge 0.$$

Then we introduce some priori estimates.

Lemma 2.2. *Let u_n and $u(t_n)$ be the solutions of (2.13) and (2.8), respectively. Assume that $u_0 \in L_2(\Omega, H^\beta)$, $0 \le \beta \le 1$, then there exists a constant $C = C(T, \|u\|, \|\sigma\|)$ such that for $t_n \in [0, T]$*

$$\|u^n\|_{L_2(\Omega;H)}^2 + \frac{\tau}{8}\sum_{j=1}^n \|\nabla(u^j + u^{j-1})\|_{L_2(\Omega;H)}^2 \le C. \tag{2.14}$$

Proof of Lemma 2.2. We take the inner product of (2.13) with $(u^n + u^{n-1})/2$ and obtain

$$\left(\frac{u^n - u^{n-1}}{\tau}, \frac{u^n + u^{n-1}}{2}\right) + \left(A\frac{u^n + u^{n-1}}{2}, \frac{u^n + u^{n-1}}{2}\right) + \left(B\left(u^{n-1}, \frac{u^n + u^{n-1}}{2}\right), \frac{u^n + u^{n-1}}{2}\right)$$
$$= \left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \frac{\sigma(u^n) + \sigma(u^{n-1})}{2}\mathrm{d}W(s), \frac{u^n + u^{n-1}}{2}\right). \tag{2.15}$$

Firstly we know that

$$\left(\frac{u^n - u^{n-1}}{\tau}, \frac{u^n + u^{n-1}}{2}\right) = \frac{1}{2\tau}(\|u^n\|^2 - \|u^{n-1}\|^2) \tag{2.16}$$

then from (2.5) we have

$$\left(B\left(u^{n-1}, \frac{u^n+u^{n-1}}{2}\right), \frac{u^n+u^{n-1}}{2}\right) = 0, \tag{2.17}$$

and from the property of operator A, we can write

$$\left(A\frac{u^n+u^{n-1}}{2}, \frac{u^n+u^{n-1}}{2}\right) = \left\|\nabla\frac{u^n+u^{n-1}}{2}\right\|^2 \geq 0. \tag{2.18}$$

The right hand term of (2.15) can be handled as

$$\left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \frac{\sigma(u^n)+\sigma(u^{n-1})}{2}\mathrm{d}W(s), \frac{u^n+u^{n-1}}{2}\right) \leq C\|\sigma(u^n)+\sigma(u^{n-1})\|_{-1}^2 + \frac{1}{2}\left\|\nabla\frac{u^n+u^{n-1}}{2}\right\|^2. \tag{2.19}$$

Taking (2.16)–(2.19) into (2.15), we derive that

$$\frac{1}{2\tau}(\|u^n\|^2 - \|u^{n-1}\|^2) + \frac{1}{4}\|\nabla(u^n+u^{n-1})\|^2 \leq C\|\sigma(u^n)+\sigma(u^{n-1})\|_{-1}^2 + \frac{1}{2}\left\|\nabla\frac{u^n+u^{n-1}}{2}\right\|^2. \tag{2.20}$$

Take the expectation on both side of (2.20), and sum up from $j = 1$ to n

$$\|u^n\|_{L_2(\Omega;H)}^2 + \frac{\tau}{8}\sum_{j=1}^{n}\|\nabla(u^j+u^{j-1})\|_{L_2(\Omega;H)}^2 = \|u_0\|^2 + \|\sigma\|_{C[0,T];H^{-1}}^2.$$

It leads to

$$\|\nabla u^n\|_{L_2(\Omega;H)} \leq C_1(T, \|u_0\|, \|\sigma\|),$$

and

$$\tau\sum_{j=1}^{n}\|\nabla(u^j+u^{j-1})\|_{L_2(\Omega;H)}^2 \leq C_2(T, \|u_0\|, \|\sigma\|).$$

The proof is now complete. □

3. The error estimate

The main results of this paper are given in the following theorem.

Theorem 3.1. *Let u_n and $u(t_n)$ be the solutions of (2.13) and (2.8), respectively. Assume that $u_0 \in L_2(\Omega, H^\beta)$, $0 \leq \beta \leq 1$, then there exists a constant $C = C(T, \|u\|, \|\sigma\|)$ such that for $t_n \in [0, T]$*

$$\|u^n - u(t_n)\|_{L_2(\Omega;H)} \leq C\left(T, \sup_{0\leq s\leq T}\|\sigma(u(s))\|_{L_2(\Omega;H)}\right)\tau^\beta.$$

From (2.12) and (2.13) we have

$$u(t_n) = E(t_n)u_0 - \int_0^{t_n} E(t_n-s)B(u(s),u(s))\mathrm{d}s + \int_0^{t_n} E(t_n-s)\sigma(u(s))\,\mathrm{d}W(s) \tag{3.21}$$

and

$$u^n = E_\tau^n u_0 - \sum_{j=1}^{n} E_\tau^{n-j}\frac{\tau}{I+\frac{\tau}{2}A}B\left(u^{j-1}, \frac{u^j+u^{j-1}}{2}\right) + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_\tau^{n-j}\left(I+\frac{\tau}{2}A\right)^{-1}\frac{\sigma(u^n)+\sigma(u^{n-1})}{2}\mathrm{d}W(s) \tag{3.22}$$

where

$$E(t) = \mathrm{e}^{-tA}, \qquad E_\tau = \frac{I-\frac{\tau}{2}A}{I+\frac{\tau}{2}A},$$

and I denotes the identity. We use t_j to denote the jth time step such that $t_j = \tau j$, $j = 0, 1, 2, \ldots, n$.

Defining error $\mathrm{e}^n = u^n - u(t_n)$, we arrive at

$$\begin{aligned}\mathrm{e}^n &= [E_\tau^n - E(t_n)]u_0 + \int_0^{t_n} E(t_n - s)B(u(s), u(s))\mathrm{d}s - \sum_{j=1}^{n} E_\tau^{n-j}\frac{\tau}{I + \frac{\tau}{2}A}B\left(u^{j-1}, \frac{u^j + u^{j-1}}{2}\right) \\ &\quad + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_\tau^{n-j}\left(I + \frac{\tau}{2}A\right)^{-1}\frac{\sigma(u^n) + \sigma(u^{n-1})}{2}\mathrm{d}W(s) - \int_0^{t_n} E(t_n - s)\sigma(u(s))\mathrm{d}W(s) \\ &=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned} \tag{3.23}$$

In order to obtain the error estimates, we split the error into three parts, denoted by $\mathrm{I}_k,\ k = 1, 2, 3$, respectively. The I_1 and I_2 are dealed similarly to the corresponding deterministic case, and I_3 is the stochastic error that we prove in a different manner.

Lemma 3.1. *Let I_1 be defined in (3.23). There exists a constant C depending on time T such that*

$$\|\mathrm{I}_1\|_{L_2(\Omega;H)} \le C\tau^\beta \|u_0\|_{L_2(\Omega;H)}.$$

Proof. We write I_1 as follow

$$\begin{aligned}\|\mathrm{I}_1\|^2_{L_2(\Omega;H)} &= \|[E_\tau^n - E(t_n)]u_0\|^2_{L_2(\Omega;H)} = \mathbf{E}\|[E_\tau^n - E(t_n)]u_0\|^2 \\ &= \mathbf{E}\left\|\left[\frac{I - \frac{\tau}{2}A}{I + \frac{\tau}{2}A}^n - \mathrm{e}^{-t_nA}\right]u_0\right\|^2,\end{aligned}$$

since we know that

$$\lim_{\tau\to 0}\frac{\mathrm{e}^{-t_nA} - \frac{I - \frac{\tau}{2}A}{I + \frac{\tau}{2}A}^n}{\tau^2} = \frac{1}{12}\mathrm{e}^{-t_nA}t_nA^3, \tag{3.24}$$

so for sufficiently small τ, we have

$$\|\mathrm{I}_1\|^2_{L_2(\Omega;H)} \le \mathbf{E}\|C(T)\tau^2 u_0\|^2 \le C\tau^{2\beta}\|u_0\|^2_{L_2(\Omega;H)}, \tag{3.25}$$

which implies that $\|\mathrm{I}_1\|_{L_2(\Omega;H)} \le C\tau^\beta\|u_0\|_{L_2(\Omega;H)}$.
The proof is now complete. □

Lemma 3.2. *Let I_2 be defined in (3.23). There exists constant C_1 and C_2 depending on time T and data $\|u\|$ on $t \in [0, T]$ such that*

$$\|\mathrm{I}_2\|^2_{L_2(\Omega;H)} \le C_1(T, \|u_0\|^2_{L_2(\Omega;H)})\tau^{2\beta} + C_2(T, \|u_0\|^2_{L_2(\Omega;H)})\tau\sum_{j=1}^{n}\mathbf{E}\|\mathrm{e}^j\|^2.$$

Proof. We write the second part I_2 as

$$\begin{aligned}\mathrm{I}_2 &= \int_0^{t_n} E(t_n - s)B(u(s), u(s))\mathrm{d}s - \sum_{j=1}^{n} E_\tau^{n-j}\frac{\tau}{I + \frac{\tau}{2}A}B\left(u^{j-1}, \frac{u^j + u^{j-1}}{2}\right) \\ &= \left[\int_0^{t_n} E(t_n - s)B(u(s), u(s))\mathrm{d}s - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\tau B\left(u(t_{j-1}), \frac{u(t_j) + u(t_{j-1})}{2}\right)\right] \\ &\quad + \left[\sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\tau B(u(t_{j-1}), \frac{u(t_j) + u(t_{j-1})}{2}) - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\tau B\left(u^{j-1}, \frac{u^j + u^{j-1}}{2}\right)\right] \\ &=: \mathrm{I}_{2,1} + \mathrm{I}_{2,2}.\end{aligned}$$

Since

$$\begin{aligned}\mathrm{I}_{2,1} &= \frac{1}{2}\left[\int_0^{t_n} E(t_n - s)B(u(s), u(s))\mathrm{d}s - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\tau B(u(t_{j-1}), u(t_j))\right] \\ &\quad + \frac{1}{2}\left[\int_0^{t_n} E(t_n - s)B(u(s), u(s))\mathrm{d}s - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\tau B(u(t_{j-1}), u(t_{j-1}))\right] \\ &=: \frac{1}{2}(\mathrm{I}_{2,1,a} + \mathrm{I}_{2,1,b}),\end{aligned} \tag{3.26}$$

and since $\mathrm{I}_{2,1,a}$, $\mathrm{I}_{2,1,b}$ can be treated in the same way, we only need to deal with $\mathrm{I}_{2,1,a}$.

$$\begin{aligned}\mathrm{I}_{2,1,a} = & \int_0^{t_n} E(t_n-s)[B(u(s),u(s)) - B(u(t_n),u(t_n))]\mathrm{d}s \\ & - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}\tau[B(u(t_{j-1}),u(t_j)) - B(u(t_n),u(t_n))] \\ & + \left[\int_0^{t_n} E(t_n-s)\mathrm{d}s - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}\tau\right]B(u(t_n),u(t_n)).\end{aligned} \tag{3.27}$$

To estimate the first part of (3.27), we split the integral into n parts

$$\begin{aligned}&\int_0^{t_n} E(t_n-s)[B(u(s),u(s)) - B(u(t_n),u(t_n))]\,\mathrm{d}s - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}\tau[B(u(t_{j-1}),u(t_j)) - B(u(t_n),u(t_n))] \\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E(t_n-s) - E(t_n-t_{j-1})][B(u(s),u(s)) - B(u(t_n),u(t_n))]\mathrm{d}s \\ &\quad + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E(t_n-t_{j-1})[B(u(s),u(s)) - B(u(t_{j-1}),u(t_j))]\mathrm{d}s \\ &\quad + \sum_{j=1}^{n}\left[E(t_n-t_{j-1}) - \frac{E_\tau^{n-j+1}}{I-\frac{\tau}{2}A}\right]\tau[B(u(t_{j-1}),u(t_j)) - B(u(t_n),u(t_n))].\end{aligned} \tag{3.28}$$

We first note that

$$\begin{aligned}E(t_n-s) - E(t_n-t_{j-1}) &= \mathrm{e}^{-(t_n-s)A} - \mathrm{e}^{-(t_n-t_{j-1})A} \\ &= \mathrm{e}^{-(t_n-s)A}(I - \mathrm{e}^{-(s-t_{j-1})A}) \\ &= E(t_n-s)(I - E(s-t_{j-1})),\end{aligned} \tag{3.29}$$

where $s \in [t_{j-1}, t_j]$.

Therefore for sufficient small τ, there exits a constant C such that

$$\begin{aligned}\|I - \mathrm{e}^{-(s-t_{j-1})A}\| &\le \|I - \mathrm{e}^{-(t_j-t_{j-1})A}\| \\ &= \|I - \mathrm{e}^{-\tau A}\| \le \|C\tau\|.\end{aligned} \tag{3.30}$$

Making use of (2.6), we have

$$\begin{aligned}\|B(u(s),u(s)) - B(u(t_n),u(t_n))\| &\le C\|Au(t)\|\|\nabla(u(s)-u(t_n))\| \\ &\le C(\|u\|)(s-t_n).\end{aligned} \tag{3.31}$$

Then by using (3.29), (3.30), (3.31) and the isometry property, we arrive at the approximation of the first line of (3.28):

$$\begin{aligned}&\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E(t_n-s) - E(t_n-t_{j-1})][B(u(s),u(s)) - B(u(t_n),u(t_n))]\mathrm{d}s\right\|^2_{L_2(\Omega;H)} \\ &= \mathbf{E}\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E(t_n-s) - E(t_n-t_{j-1})][B(u(s),u(s)) - B(u(t_n),u(t_n))]\mathrm{d}s\right\|^2 \\ &\le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|\mathbf{E}[E(t_n-s) - E(t_n-t_{j-1})][B(u(s),u(s)) - B(u(t_n),u(t_n))]\|^2\mathrm{d}s \\ &\le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|\mathbf{E}(C(T)\tau)E(t_n-s)[B(u(s),u(s)) - B(u(t_n),u(t_n))]\|^2\mathrm{d}s \\ &\le C(T)\tau^\beta\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|\mathbf{E}E(t_n-s)[B(u(s),u(s)) - B(u(t_n),u(t_n))]\|^2\mathrm{d}s \\ &\le C(T)\tau^\beta\left(\mathbf{E}\|u_0\|^2 + \sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right).\end{aligned} \tag{3.32}$$

In order to estimate the second line of (3.28) we write

$$
\begin{aligned}
&\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E(t_n-t_{j-1})[B(u(s),u(s))-B(u(t_{j-1}),u(t_j))]\mathrm{d}s\\
&\quad=\int_0^{\tau} E(t_n)B(u(s),u(s))\,\mathrm{d}s-E(t_n)\tau B(u_0,u_1)+\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} E(t_n-t_{j-1})B(u(s)-u(t_{j-1}),u(s))\mathrm{d}s\\
&\qquad+\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} E(t_n-t_{j-1})B(u(t_{j-1}),u(s)-u(t_j))\mathrm{d}s. \qquad (3.33)
\end{aligned}
$$

Since

$$\|E(t_n)B(u(s),u(s))\|\le C(T)\|u(s)\|\|\nabla u(s)\|\le C(T)(\|u_0\|+\sup_{0\le s\le T}\|u(s)\|),$$

so that

$$\left\|\int_0^{\tau} E(t_n)B(u(s),u(s))\right\|^2\le C(T,\|u\|)\tau^2.$$

Similarly,

$$\|E(t_n)B(u(s),u(s))\tau\|^2\le C(T,\|u\|)\tau^2.$$

Therefore, by using the two inequations above and the isometry property we derive the estimation of the first term of (3.33):

$$
\begin{aligned}
&\left\|\int_0^{\tau} E(t_n)B(u(s),u(s))\,\mathrm{d}s-E(t_n)\tau B(u_0,u_1)\right\|_{L_2(\Omega;H)}\\
&\quad\le\left\|\int_0^{\tau} E(t_n)B(u(s),u(s))\mathrm{d}s\right\|_{L_2(\Omega;H)}+\|E(t_n)\tau B(u_0,u_1)\|_{L_2(\Omega;H)}\\
&\quad=\left(\mathbf{E}\left\|\int_0^{\tau} E(t_n)B(u(s),u(s))\mathrm{d}s\right\|^2\right)^{1/2}+(\mathbf{E}\|E(t_n)\tau B(u_0,u_1)\|^2)^{1/2}\\
&\quad\le\left(\int_0^{\tau}\|\mathbf{E}E(t_n)\tau B(u_0,u_1)\|^2\right)^{1/2}+C(T,u_0,u_1)\tau\\
&\quad\le C\left(T,\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right)\tau.
\end{aligned}
$$

The last two terms of (3.33) can be treated in the similar manner:

$$
\begin{aligned}
&\left\|\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} E(t_n-t_{j-1})B(u(s)-u(t_{j-1}),u(s))\mathrm{d}s\right\|^2_{L_2(\Omega;H)}\\
&\quad=\mathbf{E}\left\|\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} E(t_n-t_{j-1})B(u(s)-u(t_{j-1}),u(s))\mathrm{d}s\right\|^2\\
&\quad\le C\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\|\mathbf{E}E(t_n-t_{j-1})B(u(s)-u(t_{j-1}),u(s))\|^2\mathrm{d}s\\
&\quad\le C\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}C(t_n,t_j)\|\nabla(u(s)-u(t_{j-1}))\|^2\|u(s)\|^2\mathrm{d}s\\
&\quad\le C(T)\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}(s-t_{j-1})^{\beta}\mathrm{d}s\\
&\quad\le C(T)\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\cdot n\tau^{\beta+1}\\
&\quad=C(T)\tau^{\beta}\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2,
\end{aligned}
$$

and for the last line of (3.33) we also arrive at

$$\left\|\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} E(t_n - t_{j-1})B(u(t_{j-1}), u(s) - u(t_j))\mathrm{d}s\right\|_{L_2(\Omega;H)}^2 \le C(T)\tau^{\beta} \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2.$$

Therefore we derive the approximation of (3.33)

$$\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E(t_n - t_{j-1})[B(u(s), u(s)) - B(u(t_{j-1}), u(t_j))]\mathrm{d}s\right\|_{L_2(\Omega;H)}^2 \le C\left(T, \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2\right)\tau^{\beta}. \tag{3.34}$$

The last term of (3.28) can be treated as (3.33)

$$\begin{aligned}
&\sum_{j=1}^{n}\left[E(t_n - t_{j-1}) - E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\right]\tau[B(u(t_{j-1}), u(t_j)) - B(u(t_n), u(t_n))] \\
&\quad = \left[E(t_n) - E_\tau^{n}\left(I - \frac{\tau}{2}A\right)^{-1}\right]\tau[B(u(t_0), u(t_1)) - B(u(t_n), u(t_n))] \\
&\qquad + \sum_{j=2}^{n}\left[E(t_n - t_{j-1}) - E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\right]\tau B(u(t_{j-1}) - u(t_n), u(t_j)) \\
&\qquad + \sum_{j=2}^{n}\left[E(t_n - t_{j-1}) - E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\right]\tau B(u(t_n), u(t_j) - u(t_n)),
\end{aligned} \tag{3.35}$$

and each term of (3.35) is treated in a similar manner such as

$$\begin{aligned}
&\left\|\left(E(t_n) - E_\tau^{n}\left(I - \frac{\tau}{2}A\right)^{-1}\right)\tau[B(u(t_0), u(t_1)) - B(u(t_n), u(t_n))]\right\|_{L_2(\Omega;H)} \\
&\quad = \left(\mathbf{E}\left\|\left(E(t_n) - E_\tau^{n}\left(I - \frac{\tau}{2}A\right)^{-1}\right)\tau[B(u(t_0), u(t_1)) - B(u(t_n), u(t_n))]\right\|^2\right)^{1/2} \\
&\quad \le C\left(T, \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2\right)\tau^2,
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\sum_{j=2}^{n}\left[E(t_n - t_{j-1}) - E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\right]\tau B(u(t_{j-1}) - u(t_n), u(t_j))\right\|_{L_2(\Omega;H)} \\
&\quad = \left\{\mathbf{E}\left\|\sum_{j=2}^{n}\left[E(t_n - t_{j-1}) - E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\right]\tau B(u(t_{j-1}) - u(t_n), u(t_j))\right\|^2\right\}^{1/2} \\
&\quad \le C\tau\left\{\sum_{j=2}^{n}\left[E(t_n - t_{j-1}) - E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\right]\mathbf{E}\|B(u(t_{j-1}) - u(t_n), u(t_j))\|^2\right\}^{1/2} \\
&\quad \le C\tau\left(n \sup_{0\le j\le n} C_1(t_n, t_j)\mathbf{E}\|B(u(t_{j-1}) - u(t_n), u(t_j))\|^2\right)^{1/2} \\
&\quad \le C\tau\left(C_1(T)n\tau \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2\right)^{1/2} \\
&\quad = C\left(T, \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2\right)\tau.
\end{aligned}$$

Then we arrive at the estimation of (3.35)

$$\left\|\sum_{j=1}^{n}\left[E(t_n - t_{j-1}) - \frac{E_\tau^{n-j+1}}{I - \frac{\tau}{2}A}\right]\tau[B(u(t_{j-1}), u(t_j)) - B(u(t_n), u(t_n))]\right\|_{L_2(\Omega;H)} \le C\left(T, \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2\right)\tau. \tag{3.36}$$

From (3.32), (3.34) and (3.36) we obtain

$$\begin{aligned}&\Bigg\| \int_0^{t_n} E(t_n - s)[B(u(s), u(s)) - B(u(t_n), u(t_n))]\,\mathrm{d}s \\ &\quad - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1} \tau [B(u(t_{j-1}), u(t_j)) - B(u(t_n), u(t_n))]\Bigg\|_{L_2(\Omega;H)} \\ &\le C\left(T, \sup_{0\le s\le T} \mathbf{E}\|u(s)\|^2\right)\tau^{2\beta}. \end{aligned}\tag{3.37}$$

Then we consider the second part of (3.27). It is easy to verify that

$$\int_0^{t_n} E(t_n - s)\,\mathrm{d}s = \int_0^{t_n} \mathrm{e}^{-(t_n-s)A}\mathrm{d}s = \mathrm{e}^{-t_nA}\int_0^{t_n}\mathrm{e}^{sA}\,\mathrm{d}s = \mathrm{e}^{-t_nA}(A^{-1}\mathrm{e}^{t_nA} - A^{-1}) = (I - E(t_n))A^{-1},$$

and

$$\sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\tau = \frac{E_\tau - E_\tau^{n+1}}{I - E_\tau}\left(I - \frac{\tau}{2}A\right)^{-1}\tau = (I - E_\tau^n)A^{-1}.$$

So we have that

$$\begin{aligned}&\left\|\left[\int_0^{t_n} E(t_n - s)\mathrm{d}s - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\tau\right] B(u(t_n), u(t_n))\right\|_{L_2(\Omega;H)} \\ &= (\mathbf{E}\|(E_\tau^n - E(t_n))A^{-1}B(u(t_n), u(t_n))\|^2)^{1/2} \\ &\le C(t_n)\tau^2(\mathbf{E}\|B(u(t_n), u(t_n))\|^2)^{1/2} \\ &\le C\left(T, \sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right)\tau^2. \end{aligned}\tag{3.38}$$

Eqs. (3.37) and (3.38) yield

$$\|\mathrm{I}_{2,1,a}\|_{L_2(\Omega;H)} \le C\left(T, \sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right)\tau^\beta.\tag{3.39}$$

Similarly we have the same approximation for $\mathrm{I}_{2,1,b}$,

$$\|\mathrm{I}_{2,1,b}\|_{L_2(\Omega;H)} \le C\left(T, \sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right)\tau^\beta.\tag{3.40}$$

Therefore,

$$\begin{aligned}\|\mathrm{I}_{2,1}\|_{L_2(\Omega;H)} &\le C(\|\mathrm{I}_{2,1,a}\|_{L_2(\Omega;H)} + \|\mathrm{I}_{2,1,b}\|_{L_2(\Omega;H)}) \\ &\le C\left(T, \sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right)\tau^\beta. \end{aligned}\tag{3.41}$$

Then we deal with $\mathrm{I}_{2,2}$,

$$\begin{aligned}\mathrm{I}_{2,2} &= \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\tau B\left(u(t_{j-1}), \frac{u(t_j) + u(t_{j-1})}{2}\right) - \sum_{j=1}^{n} E_\tau^{n-j+1}\left(I - \frac{\tau}{2}A\right)^{-1}\tau B\left(u^{j-1}, \frac{u^j + u^{j-1}}{2}\right) \\ &= \frac{1}{2I - \tau A}\left\{\left[\sum_{j=1}^{n} E_\tau^{n-j+1}\tau B(u(t_{j-1}), u(t_j)) - \sum_{j=1}^{n} E_\tau^{n-j+1}\tau B(u^{j-1}, u^j)\right]\right. \\ &\quad \left. + \left[\sum_{j=1}^{n} E_\tau^{n-j+1}\tau B(u(t_{j-1}), u(t_{j-1})) - \sum_{j=1}^{n} E_\tau^{n-j+1}\tau B(u^{j-1}, u^{j-1})\right]\right\} \\ &=: \frac{1}{2I - \tau A}(\mathrm{I}_{2,2,a} + \mathrm{I}_{2,2,b}). \end{aligned}$$

Again, we only need to consider $\mathrm{I}_{2,2,a}$, since $u^0 = u(t_0) = u_0$

$$\begin{aligned}
\mathrm{I}_{2,2,a} &= \sum_{j=1}^{n} E_\tau^{n-j+1}\tau B(u(t_{j-1}), u(t_j)) - \sum_{j=1}^{n} E_\tau^{n-j+1}\tau B(u^{j-1}, u^j) \\
&= \sum_{j=1}^{n} E_\tau^{n-j+1}\tau [B(u(t_{j-1}), u(t_j)) - B(u^{j-1}, u^j)] \\
&= \sum_{j=1}^{n} E_\tau^{n-j+1}\tau B(u(t_{j-1}) - u^{j-1}, u(t_j)) + \sum_{j=1}^{n} E_\tau^{n-j+1}\tau B(u^{j-1}, u(t_j) - u^j) \\
&= \sum_{j=1}^{n-1} E_\tau^{n-j}\tau B(u(t_j) - u^j, u(t_{j+1})) + \sum_{j=1}^{n-1} E_\tau^{n-j+1}\tau B(u^{j-1}, u(t_j) - u^j) + E_\tau \tau B(u^{n-1}, u(t_n) - u^n).
\end{aligned}$$

The three terms of $\mathrm{I}_{2,2,a}$ can be treated in a same manner, since we know from (2.6)

$$\begin{aligned}
\left\| \sum_{j=1}^{n-1} E_\tau^{n-j}\tau B(u(t_j) - u^j, u(t_{j+1})) \right\|^2_{L_2(\Omega;H)} &= \mathbf{E} \left\| \sum_{j=1}^{n-1} E_\tau^{n-j}\tau B(u(t_j) - u^j, u(t_{j+1})) \right\|^2 \\
&\le C\tau \sum_{j=1}^{n-1} \mathbf{E}\|B(u(t_j) - u^j, u(t_{j+1}))\|^2 \\
&\le C\tau \sum_{j=1}^{n-1} \mathbf{E}\|u(t_j) - u^j\|^2 \|u(t_{j+1})\|^2 \\
&\le C(T, \|u\|)\tau \sum_{j=1}^{n-1} \mathbf{E}\|\mathrm{e}^j\|^2,
\end{aligned}$$

which implies that

$$\|\mathrm{I}_{2,2,a}\|^2_{L_2(\Omega;H)} \le C(T, \|u\|)\tau \sum_{j=1}^{n} \mathbf{E}\|\mathrm{e}^j\|^2.$$

Therefore we have

$$\|\mathrm{I}_{2,2}\|^2_{L_2(\Omega;H)} \le C(\|\mathrm{I}_{2,2,a}\|^2_{L_2(\Omega;H)} + \|\mathrm{I}_{2,2,b}\|^2_{L_2(\Omega;H)}) \le C(T, \|u\|)\tau \sum_{j=1}^{n} \mathbf{E}\|\mathrm{e}^j\|^2. \tag{3.42}$$

From (3.41) and (3.42)

$$\|\mathrm{I}_2\|^2_{L_2(\Omega;H)} \le C_1\tau^{2\beta} + C_2\tau \sum_{j=1}^{n} \mathbf{E}\|\mathrm{e}^j\|^2$$

which complete the proof. □

Lemma 3.3. *Let* I_3 *be defined in* (3.23)*. There exists constant* C_1 *and* C_2 *depending on time* T *and data* $\|u\|$ *on* $t \in [0, T]$ *such that*

$$\|\mathrm{I}_3\|^2_{L_2(\Omega;H)} \le C_1\left(T, \sup_{0\le s\le T} \mathbf{E}\|\sigma(u(s))\|^2\right)\tau^{2\beta} + C_2\tau \sum_{j=1}^{n} \mathbf{E}\|\mathrm{e}^j\|^2. \tag{3.43}$$

Proof. For I_3, we have

$$\begin{aligned}
\mathrm{I}_3 &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E_\tau^{n-j}\left(I + \frac{\tau}{2}A\right)^{-1} \frac{\sigma(u^n) + \sigma(u^{n-1})}{2}\mathrm{d}W(s) - \int_0^{t_n} E(t_n - s)\sigma(u(s))\mathrm{d}W(s) \\
&= \frac{1}{2}\left\{\left[\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E_\tau^{n-j}\left(I + \frac{\tau}{2}A\right)^{-1} \sigma(u^{n-1})\mathrm{d}W(s) - \int_0^{t_n} E(t_n - s)\sigma(u(s))\mathrm{d}W(s)\right]\right.
\end{aligned}$$

$$+\left[\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_\tau^{n-j}\left(I+\frac{\tau}{2}A\right)^{-1}\sigma(u^n)\mathrm{d}W(s)-\int_0^{t_n}E(t_n-s)\sigma(u(s))\mathrm{d}W(s)\right]\Bigg\}$$
$$=: \frac{1}{2}(\mathrm{I}_{3,a}+\mathrm{I}_{3,b}). \tag{3.44}$$

Here we only need to consider $\mathrm{I}_{3,a}$ since the approximation of $\mathrm{I}_{3,b}$ can be obtain by the same method.
For $\mathrm{I}_{3,a}$, we split it into four parts, and deal with each part separately.

$$\begin{aligned}
\mathrm{I}_{3,a} &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}\sigma(u^{j-1})\mathrm{d}W(s)-\int_0^{t_n}E(t_n-s)\sigma(u(s))\mathrm{d}W(s)\\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}[\sigma(u^{j-1})-\sigma(u(t_{j-1}))]\mathrm{d}W(s)\\
&\quad+\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}[\sigma(u(t_{j-1}))-\sigma(u(s))]\mathrm{d}W(s)\\
&\quad+\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left[E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}-E(t_n-t_{j-1})\right]\sigma(u(s))\mathrm{d}W(s)\\
&\quad+\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}[E(t_n-t_{j-1})-E(t_n-s)]\sigma(u(s))\mathrm{d}W(s)\\
&=: \mathrm{I}_{3,a,1}+\mathrm{I}_{3,a,2}+\mathrm{I}_{3,a,3}+\mathrm{I}_{3,a,4}.
\end{aligned}\tag{3.45}$$

By isometry property (2.11) and Lipschitz conditon (2.7), we have the following result for $\mathrm{I}_{3,a,1}$,

$$\begin{aligned}
\|\mathrm{I}_{3,a,1}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}[\sigma(u^{j-1})-\sigma(u(t_{j-1}))]\mathrm{d}W(s)\right\|^2\\
&= \tau\sum_{j=1}^{n}\mathbf{E}\left\|E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}[\sigma(u^{j-1})-\sigma(u(t_{j-1}))]\right\|^2\\
&\le C\tau\sum_{j=1}^{n}\mathbf{E}\|\sigma(u^{j-1})-\sigma(u(t_{j-1}))\|^2\le C\tau\sum_{j=1}^{n}\mathbf{E}\|u^{j-1}-u(t_{j-1})\|^2\\
&= C\tau\sum_{j=1}^{n}\mathbf{E}\|\mathrm{e}^{j-1}\|^2 = C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\|\mathrm{e}^{j-1}\|^2\mathrm{d}s\le C\tau\sum_{j=1}^{n}\mathbf{E}\|\mathrm{e}^{j}\|^2.
\end{aligned}\tag{3.46}$$

For $\mathrm{I}_{3,a,2}$, we can deal similarly to (3.46)

$$\begin{aligned}
\|\mathrm{I}_{3,a,2}\|^2_{L_2(\Omega;H)} &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}[\sigma(u(t_{j-1}))-\sigma(u(s))]\right\|^2\mathrm{d}s\\
&\le C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\|\sigma(u(t_{j-1}))-\sigma(u(s))\|^2\mathrm{d}s\\
&\le C\left(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(s-t_{j-1})^{\beta}\mathrm{d}s\right)\left(\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right)\\
&\le C\left(\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right)\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\tau^{2\beta}\mathrm{d}s\\
&= C\left(T,\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right)\tau^{2\beta}.
\end{aligned}\tag{3.47}$$

In order to estimate $\mathrm{I}_{3,a,3}$ and $\mathrm{I}_{3,a,4}$, we only need to focus on the operator E and E_τ. Since for every $j=1,2,\ldots,n$, we have the inequation from (3.24)

$$E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}-E(t_n-t_{j-1})\le C(T,j)\tau^2,$$

which follows that

$$\begin{aligned}\|\mathrm{I}_{3,a,3}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left(E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}-E(t_n-t_{j-1})\right)\sigma(u(s))\mathrm{d}W(s)\right\|^2\\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|(E_\tau^{n-j+1}\left(I-\frac{\tau}{2}A\right)^{-1}-E(t_n-t_{j-1}))\sigma(u(s))\right\|^2\mathrm{d}s\\ &\le C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\tau^{2\beta}\sup_{t_{j-1}\le s\le t_j}\mathbf{E}\|\sigma(u(s))\|^2\mathrm{d}s\\ &\le C\left(\sup_{0\le s\le T}\mathbf{E}\|\sigma(u(s))\|^2\right)\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\tau^{2\beta}\mathrm{d}s\\ &= C\left(T,\sup_{0\le s\le T}\mathbf{E}\|\sigma(u(s))\|^2\right)\tau^{2\beta}.\end{aligned} \tag{3.48}$$

Similarly, from (3.29) and (3.30) we derive

$$\begin{aligned}\|\mathrm{I}_{3,a,4}\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(E(t_n-t_{j-1})-E(t_n-s))\sigma(u(s))\mathrm{d}W(s)\right\|^2\\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\|E(t_n-s)(I-E(s-t_{j-1}))\sigma(u(s))\|^2\mathrm{d}s\\ &\le C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\tau^{2\beta}\sup_{t_{j-1}\le s\le t_j}\mathbf{E}\|\sigma(u(s))\|^2\mathrm{d}s\\ &= C\left(T,\sup_{0\le s\le T}\mathbf{E}\|\sigma(u(s))\|^2\right)\tau^{2\beta}.\end{aligned} \tag{3.49}$$

Then we can obtain the estimation of $\mathrm{I}_{3,b,1}\sim\mathrm{I}_{3,b,4}$, since they are same to $\mathrm{I}_{3,a,1}\sim\mathrm{I}_{3,a,4}$. Together these estimations we obtain (3.43).

Now we complete the proof of the main theorem. □

Proof of Theorem 3.1. From Lemmas 3.1–3.3 we arrive at

$$\begin{aligned}\mathbf{E}\|\mathrm{e}^n\|^2 &= \|\mathrm{I}_1\|^2_{L_2(\Omega;H)}+\|\mathrm{I}_2\|^2_{L_2(\Omega;H)}+\|\mathrm{I}_3\|^2_{L_2(\Omega;H)}\\ &= \|\mathrm{I}_1\|^2_{L_2(\Omega;H)}+\sum_{k=1}^{2}\|\mathrm{I}_{2,k}\|^2_{L_2(\Omega;H)}+\sum_{k=1}^{4}\|\mathrm{I}_{3,a,k}\|^2_{L_2(\Omega;H)}+\sum_{k=1}^{4}\|\mathrm{I}_{3,b,k}\|^2_{L_2(\Omega;H)}\\ &\le C_1\left(T,\sup_{0\le s\le T}\mathbf{E}\|\sigma(u(s))\|^2\right)\tau^{2\beta}+C_2\left(T,\sup_{0\le s\le T}\mathbf{E}\|\sigma(u(s))\|^2\right)\tau\sum_{j=1}^{n}\mathbf{E}\|\mathrm{e}^j\|^2.\end{aligned}$$

By the discrete Gronwall lemma, we get

$$\mathbf{E}\|\mathrm{e}^n\|^2\le C\left(T,\sup_{0\le s\le T}\mathbf{E}\|\sigma(u(s))\|^2\right)\tau^{2\beta},$$

which implies that

$$\|\mathrm{e}^n\|_{L_2(\Omega;H)}\le C\left(T,\sup_{0\le s\le T}\|\sigma(u(s))\|_{L_2(\Omega;H)}\right)\tau^{\beta}.$$

The proof is now complete. □

4. Conclusion

In this paper, we derive the regularity property and error estimate of the stochastic Navier–Stokes equation. First, we introduce the filtered probability space. Then we define the L_2 space and its norm. Under the L_2 norm, we give several inequalities, such as the property of operators A and B, the Lipschitz condition of σ, and the isometry property about the expectation $\mathbf{E}$. Before we start to dicuss our main result, some preliminary results are proved. In the main Theorem 3.1 we give the optimal strong convergence error estimates in the L_2 norm. The proof of the main theorem is based on appropriate error estimates for the corresponding deterministic Navier–Stokes equation.

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